

# CONJUGACY CLASSES OF CENTRALIZERS IN UNITARY GROUPS

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**ABSTRACT.** Let  $G$  be a group. Two elements  $x, y \in G$  are said to be in the same  $z$ -class if their centralizers in  $G$  are conjugate within  $G$ . Consider  $\mathbb{F}$  a perfect field of characteristic  $\neq 2$  which has a non-trivial Galois automorphism of order 2. Further, suppose that the fixed field  $\mathbb{F}_0$  has the property that it has only finitely many field extensions of any finite degree. In this paper, we prove that the number of  $z$ -classes in unitary group over such fields is finite. Further, we count the number of  $z$ -classes in the finite unitary group  $U_n(q)$  and prove that this number is same as that of  $GL_n(q)$ .

## 1. INTRODUCTION

Let  $G$  be a group. Two elements  $x$  and  $y \in G$  are said to be  $z$ -equivalent, denoted as  $x \sim_z y$ , if their centralizers in  $G$  are conjugate, i.e.,  $Z_G(y) = gZ_G(x)g^{-1}$  for some  $g \in G$  where  $Z_G(x)$  denotes centralizer of  $x$  in  $G$ . Clearly  $\sim_z$  is an equivalence relation on  $G$ . The equivalence classes with respect to this relation are called  $z$ -classes. It is easy to see that if two elements of a group  $G$  are conjugate then their centralizers are conjugate thus they are also  $z$ -equivalent. However the converse need not be true, in general. We are interested in groups of Lie type where a group may have infinitely many conjugacy classes but finitely many  $z$ -classes. In geometry,  $z$ -classes describe the behavior of dynamical types. That is, if a group  $G$  is acting on a manifold  $M$  then understanding (dynamical types of) orbits is related to understanding (conjugacy classes of) centralizers. In this paper, we explore this topic for certain classical groups. Steinberg [St] (section 3.6 Corollary 1 to Theorem 2) proved that for a reductive algebraic group  $G$  defined over an algebraically closed field, of good characteristic, the number of  $z$ -classes are finite (even though there could be infinitely many conjugacy classes). Thus, it is natural to ask how far “finiteness of  $z$ -classes” holds true for algebraic groups defined over a base field. This is certainly not true even for  $GL_2$  over field  $\mathbb{Q}$  (see section 3.3). Thus we need to restrict to certain kind of fields.

**Definition 1.1** (Property FE). *A perfect field  $\mathbb{F}$  of characteristic  $\neq 2$  has the property FE if  $\mathbb{F}$  has only finitely many field extensions of any finite degree.*

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Examples of such fields are algebraically closed field (for example,  $\mathbb{C}$ ), real numbers  $\mathbb{R}$ , local fields (e.g.,  $p$ -adics  $\mathbb{Q}_p$ ) and finite fields  $\mathbb{F}_q$ . In [Ku] for  $GL_n$  and in [GK1] for orthogonal groups  $O(V, B)$  and symplectic groups  $Sp(V, B)$ , it is proved that over a field with property FE these groups have only finitely many  $z$ -classes. In this article, we extend this result to the unitary groups. We prove the following result,

**Theorem 1.2.** *Let  $\mathbb{F}$  be a field with a non-trivial Galois automorphism of order 2 and fixed field  $\mathbb{F}_0$ . Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  with a non-degenerate hermitian form  $B$ . Suppose that the fixed field  $\mathbb{F}_0$  has the property FE. Then, the number of  $z$ -classes in the unitary group  $U(V, B)$  is finite.*

This theorem is proved in section 3.2.

If we look at the character table of  $SL_2(q)$  (for example see [B] or [Pr]) we notice that the conjugacy classes and irreducible characters are grouped together. One observes a similar pattern in the work of Srinivasan [Sr] for  $Sp_4(q)$ . In [Gr], Green studied the complex representations of  $GL_n(q)$  where he introduced the function  $t(n)$  for the ‘types of characters/classes’ (towards the end of section 1 on page 407-408) which is the number of  $z$ -classes in  $GL_n(q)$ . He observed, we quote from Green (page 408 in [Gr]), the following:

The number  $t(n)$  appears as the number of rows or columns of a character table of  $GL_n(q)$ . This is because the irreducible characters, which by a well known theorem of representation theory are the same in number as the conjugacy classes, themselves collect into types in a corresponding way, and the values of all the characters of a given type at all the classes of a given type can be included in a single functional expression.

In Deligne-Lusztig theory, one studies the representation theory of finite groups of Lie type and the  $z$ -classes of semisimple elements in the dual group play an important role. In [Hu] Humphreys (Section 8.11) defined genus of an element in algebraic group  $G$  over  $k$ . Two elements have same genus if they are  $z$ -equivalent in  $G(k)$  and the genus number (respectively semisimple genus number) is the number of  $z$ -classes (respectively the number of  $z$ -classes of semisimple elements). Thus understanding  $z$ -classes for finite groups of Lie type, specially semisimple genus, and their counting is of importance in representation theory (see [Ca, DM, Fl, FG]). Bose, in [Bo], calculated genus number for simply connected simple algebraic groups over algebraically closed field and compact simple Lie groups. Classification of  $z$ -classes in  $U(n, 1)$ , the isometry group of the  $n$ -dimensional complex hyperbolic space is done in [CG]. The second named author studied  $z$ -classes for the compact group of type  $G_2$  in [Si]. In this paper we count the  $z$ -classes in complex hyperbolic groups  $U(n, 1)$  and in finite unitary groups. The following theorem for finite unitary groups has been anticipated by representation theorists (Section 4.4):

**Theorem 1.3.** *The number of  $z$ -classes in  $U_n(q)$  is same as the number of  $z$ -classes in  $GL_n(q)$ . Thus they have same generating function.*

## 2. CONJUGACY CLASSES AND CENTRALIZERS IN UNITARY GROUPS

In this section, we introduce the unitary groups. For details, reader can consult Grove [Gv] or any other textbook on the subject of classical groups. Let  $\mathbb{F}$  be a field with a non-trivial Galois automorphism  $\sigma$  of order 2 which, to simplify notation, we denote by  $\bar{a} = \sigma(a)$ . The sub-field of  $\mathbb{F}$  fixed by  $\sigma$  is denoted by  $\mathbb{F}_0$ . Let  $V$  be a vector space over  $\mathbb{F}$ . A hermitian form on  $V$  is a map  $B: V \times V \rightarrow \mathbb{F}$  which is linear in the first coordinate and satisfies  $B(u, v) = \overline{B(v, u)}$  for all  $u, v \in V$ . We consider only non-degenerate hermitian forms. An isometry of  $(V, B)$  is called a unitary transformation and the set of all such transformations is called **unitary group**  $U(V, B) = \{g \in GL(V) \mid B(u, v) = B(g(u), g(v)), \forall u, v \in V\}$ . There could be more than one hermitian form up to equivalence over a given field and similarly more than one non-isomorphic unitary group. We discuss some of the cases in Section 4.

To study  $z$ -classes, it is important to understand the conjugacy classes first. This has been well understood for classical groups through the work of Asai, Ennola, Macdonald, Milnor, Springer-Steinberg, Wall, Williamson [As, En, Ma, Mi, SS, Wa, Wi] and many others. Since the unitary group is a subgroup of  $GL_n(\mathbb{F})$  one hopes to exploit the theory of canonical forms to get the conjugacy classes in the unitary group. For our exposition we follow Springer-Steinberg [SS]. We begin with recalling notation involved in the description of conjugacy classes.

**2.1. Self-U-reciprocal Polynomials.** Let  $f(x) = \sum_{i=0}^d a_i x^i$  be a polynomial in  $\mathbb{F}[x]$  of degree  $d$ . We extend the involution on  $\mathbb{F}$  to that of  $\mathbb{F}[x]$  by  $\overline{f(x)} := \sum_{i=0}^d \bar{a}_i x^i$ . Let  $f(x)$  be a polynomial with  $f(0) \neq 0$ . The corresponding U-reciprocal polynomial of  $f(x)$  is defined by

$$\tilde{f}(x) := \overline{f(0)^{-1}} x^d \overline{f(x^{-1})}.$$

A monic polynomial  $f(x)$  with non-zero constant term is said to be **self-U-reciprocal** if  $f(x) = \tilde{f}(x)$ . In terms of roots it means that for a self-U-reciprocal polynomial whenever  $\lambda$  is a root  $\bar{\lambda}^{-1}$  is also a root with the same multiplicity. Note that  $f(x) = \tilde{\tilde{f}}(x)$ , and if  $f(x) = f_1(x)f_2(x)$  then  $\tilde{f}(x) = \tilde{f}_1(x)\tilde{f}_2(x)$ . Also  $f(x)$  is irreducible if and only if  $\tilde{f}(x)$  is irreducible. In the case when  $f(x) = (x - \lambda)^n$  the polynomial  $f(x)$  is self-U-reciprocal if and only if  $\lambda\bar{\lambda} = 1$ . Over a finite field, we have the following result due to Ennola (Lemma 2 [En]):

**Proposition 2.1.** *Let  $f(x)$  be a monic, irreducible, self-U-reciprocal polynomial over a finite field  $\mathbb{F}_{q^2}$ . Then the degree of  $f(x)$  is odd.*

Let  $T \in GL(V)$  with its minimal polynomial  $f(x)$  then  $\tilde{f}(x)$  is minimal polynomial of  $\bar{T}^{-1}$ . If  $T \in U(V, B)$  then its minimal polynomial is monic with non zero constant term and is self-U-reciprocal. Let  $f(x)$  be a self-U-reciprocal polynomial. We can write it as follows,

$$(2.1) \quad f(x) = \prod_{i=1}^{k_1} p_i(x)^{m_i} \prod_{j=1}^{k_2} (q_j(x)\tilde{q}_j(x))^{n_j}$$

where  $p_i(x)$  and  $q_j(x)$  are irreducible and  $p_i(x)$  are self-U-reciprocal but  $q_j(x)$  are not self-U-reciprocal.

**2.2. Space Decomposition with respect to a Unitary Transformation.** Let  $T \in U(V, B)$  with minimal polynomial  $f(x)$ . Write  $f(x) = \prod_i f_i(x)^{s_i}$  as in the equation 2.1. Then,

**Proposition 2.2.** *The direct sum decomposition  $V = \bigoplus_i \ker(f_i(T)^{s_i})$  is a decomposition into non-degenerate mutually orthogonal  $T$ -invariant subspaces.*

This decomposition helps us reduce the questions about conjugacy classes and  $z$ -classes of a unitary transformation to the unitary transformations with minimal polynomial of one of the following two kinds:

- Type 1.  $p(x)^m$  where  $p(x)$  is a monic, irreducible, self-U-reciprocal polynomial with non-zero constant term,
- Type 2.  $(q(x)\tilde{q}(x))^m$  where  $q(x)$  is a monic, irreducible, not self-U-reciprocal polynomial with non-zero constant term.

Thus proposition gives us a primary decomposition of  $V$  into  $T$ -invariant  $B$ -non-degenerate subspaces

$$(2.2) \quad V = \left( \bigoplus_{i=1}^{k_1} V_i \right) \oplus \left( \bigoplus_{j=1}^{k_2} V_j \right)$$

where  $V_i$  corresponds to the polynomials of Type 1 and  $V_j = V_{q_j} + V_{\tilde{q}_j}$  corresponds to the polynomials of Type 2. Denote the restriction of  $T$  to each  $V_r$  by  $T_r$ . Then the minimal polynomial of  $T_r$  is one of the two types. It turns out that the centralizer of  $T$  in  $U(V, B)$  is

$$\mathcal{Z}_{U(V, B)}(T) = \prod_r \mathcal{Z}_{U(V_r, B_r)}(T_r)$$

where  $B_r$  is hermitian form obtained by restricting  $B$  to  $V_r$ . Thus the conjugacy class and the  $z$ -class of  $T$  is determined by the restriction of  $T$  to each of the primary subspaces. Hence it is enough to determine the conjugacy and the  $z$ -class of  $T \in U(V, B)$  which has minimal polynomial of one of the types.

**2.3. Conjugacy Classes and Centralizers.** Now define  $E^T = \frac{\mathbb{F}[x]}{\langle f(x) \rangle}$  which is a  $\mathbb{F}$  algebra. Then  $V$  becomes an  $E^T$ -module where  $x$  acts via  $T$ . To keep track of the action we denote this module by  $V^T$ . The  $E^T$  module structure on  $V^T$  determines  $GL$ -conjugacy class of  $T$ . To determine a conjugacy class within  $U(V, B)$ , Springer and Steinberg (see 2.6 in [SS] Chapter 4) defined a hermitian form  $H^T$  on  $V^T$ . Since  $f(x)$  is self-U-reciprocal, there exists a unique involution  $\alpha$  on  $E^T$  such that  $\alpha(x) = x^{-1}$  and  $\alpha$  is an extension of  $\sigma$  on scalars. Thus  $(E^T, \alpha)$  is an algebra with involution. They prove that there exists a  $\mathbb{F}$ -linear function  $l: E^T \rightarrow \mathbb{F}$  such that the symmetric bilinear form  $\bar{l}: E^T \times E^T \rightarrow \mathbb{F}$  given by  $\bar{l}(a, b) = l(ab)$  is non-degenerate with  $l(\alpha(a)) = l(a)$  for all  $a$ . Further the hermitian form  $H^T$  on  $E^T$ -module  $V^T$  (with respect to  $\alpha$ ) satisfies  $B(eu, v) = l(eH^T(u, v))$  for all  $e \in E^T$  and  $u, v \in V^T$ . They prove (see 2.7 and 2.8 [SS] Chapter 4),

**Proposition 2.3.** *With notation as above, let  $S$  and  $T \in U(V, B)$ . Then,*

- (1) *the elements  $S$  and  $T$  are conjugate in  $U(V, B)$  if and only if*
  - (a) *there exists an isomorphism  $\psi: E^S \rightarrow E^T$  mapping  $x$  to  $x$  and*
  - (b) *a  $E^S$ -module isomorphism  $\phi: V^S \rightarrow V^T$  which makes the hermitian forms  $H^S$  and  $H^T$  equivalent on  $V$ .*
- (2) *The centralizer  $\mathcal{Z}_{U(V, B)}(T) = U(V^T, H^T)$ .*

We can decompose  $E^T = E_1 \oplus E_2 \oplus \cdots \oplus E_r$  where each  $E_i$  is indecomposable with respect to  $\alpha$  (see section 2.2 Chapter 4 of [SS]). The restriction of  $\alpha$  to  $E_i$  is an involution on  $E_i$  denoted as  $\alpha_i$ . Clearly  $E_i$  are of the following type according to the decomposition of  $f(x)$  in the equation 2.1.

- $\frac{\mathbb{F}[x]}{\langle p(x)^d \rangle}$  where  $p(x)$  is a monic, irreducible, self-U-reciprocal polynomial.
- $\frac{\mathbb{F}[x]}{\langle q(x)^d \rangle} \oplus \frac{\mathbb{F}[x]}{\langle \bar{q}(x)^d \rangle}$ , where  $q(x)$  is monic, irreducible but not self-U-reciprocal.

In the second case, the two components  $\frac{\mathbb{F}[x]}{\langle q(x)^d \rangle}$  and  $\frac{\mathbb{F}[x]}{\langle \bar{q}(x)^d \rangle}$  are isomorphic local rings and the restriction of  $\alpha$  is given by  $\alpha(a, b) = (b, a)$  via the isomorphism. Using Wall's approximation theorem (see Corollary 2.5 in the next section) it is easy to see that all hermitian forms over such rings are equivalent. Thus to determine equivalence of  $H^T$  we need to look at modules over rings of the first Type above.

**2.4. Wall's Approximation Theorem.** We have reduced the conjugacy problem to equivalence of hermitian forms over certain rings. We recall a theorem of Wall (see [Wa] Theorem 2.2.1) which would be useful for further analysis. Let  $R$  be a commutative ring with 1 and  $J$  be its Jacobson radical and  $\alpha$  be an involution on  $R$ . Let  $(M, B)$  be any non-degenerate hermitian space of rank  $n$  over  $R$ . We define  $\underline{M} := \frac{M}{JM}$  a module over  $\underline{R} := \frac{R}{J}$ . Now  $B$  induces a hermitian form  $\underline{B}$  on  $\underline{M}$  with respect to the involution  $\underline{\alpha}$  of  $\underline{R}$  induced by  $\alpha$ .

**Theorem 2.4** (Wall's Approximation Theorem). *With notation as above,*

- (1) *any non-degenerate hermitian form over  $\underline{R}$  is induced by some non-degenerate hermitian form over  $R$ .*
- (2) *Let  $(M_1, B_1)$  and  $(M_2, B_2)$  be non-degenerate hermitian spaces over  $R$  and correspondingly  $(\underline{M}_1, \underline{B}_1)$  and  $(\underline{M}_2, \underline{B}_2)$  be non-degenerate hermitian spaces over  $\underline{R}$ . Then  $(M_1, B_1)$  is equivalent to  $(M_2, B_2)$  if and only if  $(\underline{M}_1, \underline{B}_1)$  is equivalent to  $(\underline{M}_2, \underline{B}_2)$ .*

For our purpose we need the following,

**Corollary 2.5.** *Let  $V$  be a module over  $A = \frac{\mathbb{F}[x]}{\langle q(x) \rangle} \oplus \frac{\mathbb{F}[x]}{\langle \bar{q}(x) \rangle}$  and  $H_1$  and  $H_2$  be two non-degenerate hermitian forms on  $V$  with respect to the “switch” involution on  $A$  given by  $\overline{(b, a)} = (a, b)$ . Then  $H_1$  and  $H_2$  are equivalent.*

*Proof.* We use Wall's Approximation Theorem. Here  $R = A$  and its Jacobson radical is  $J = \frac{\langle q(x) \rangle}{\langle q(x) \rangle} \oplus \frac{\langle \bar{q}(x) \rangle}{\langle \bar{q}(x) \rangle}$ . Then  $\underline{R} \cong \frac{\mathbb{F}[x]}{\langle q(x) \rangle} \oplus \frac{\mathbb{F}[x]}{\langle \bar{q}(x) \rangle} \cong K \oplus K$  where  $K \cong \frac{\mathbb{F}[x]}{\langle q(x) \rangle} \cong \frac{\mathbb{F}[x]}{\langle \bar{q}(x) \rangle}$  is a finite extension of  $\mathbb{F}$  (thus separable). Now we have hermitian forms  $\underline{H}_i: \underline{V} \times \underline{V} \rightarrow \underline{A}$  defined by  $\underline{H}_i(u + JV, v + JV) = H_i(u, v)J$  for all  $u, v \in V$ . Thus it is enough to show that  $\underline{H}_1$  is equivalent to  $\underline{H}_2$  on  $K \oplus K$ -module  $\underline{V}$ . The norm map  $N: (K \oplus K)^\times \rightarrow K^\times$  is  $N(a, b) = \overline{(a, b)}(a, b) = (b, a)(a, b) = (ab, ab)$ . Clearly this norm map is surjective. Thus  $\frac{K^\times}{\text{Im}(N)}$  is trivial. Hence the hermitian form is unique in this case.  $\square$

**2.5. Unipotent Elements.** We look at the Type 1 more closely where the minimal polynomial is  $p(x)^d$  with  $p(x)$  an irreducible, self-U-reciprocal polynomial. This includes unipotent elements. The theory of rational canonical forms gives a decomposition of  $V = \bigoplus_{i=1}^k V_{d_i}$  with  $1 \leq d_1 \leq d_2 \leq \dots \leq d_k = d$  and each  $V_{d_i}$  is a free module over  $\mathbb{F}$ -algebra  $\frac{\mathbb{F}[x]}{\langle p(x)^{d_i} \rangle}$  (see 2.14 Chapter 4 [SS]). Thus,

**Proposition 2.6.** *Let  $S$  and  $T$  be in  $U(V, B)$ . Suppose the minimal polynomial of both  $S$  and  $T$  are equal and it is  $p(x)^d$ , where  $p(x)$  is irreducible self-U-reciprocal. Then  $S$  and  $T$  are conjugate in  $U(V, B)$  if and only if*

- (1) *the elementary divisors  $p(x)^{d_i}$  where  $1 \leq d_1 \leq d_2 \leq \dots \leq d_k = d$  of  $S$  and  $T$  are same and*
- (2) *the sequence of hermitian spaces,  $\{(V_{d_1}^S, H_{d_1}^S), \dots, (V_{d_k}^S, H_{d_k}^S)\}$  corresponding to  $S$  and  $\{(V_{d_1}^T, H_{d_1}^T), \dots, (V_{d_k}^T, H_{d_k}^T)\}$  corresponding to  $T$  are equivalent. Here  $H_{d_i}^S$  and  $H_{d_i}^T$  take values in the cyclic  $\mathbb{F}$ -algebra  $\frac{\mathbb{F}[x]}{\langle p(x)^{d_i} \rangle}$ .*

Further, the centralizer of  $T$ , in this case, is the direct product  $\mathcal{Z}_{U(V, B)}(T) = \prod_{i=1}^k U(V_{d_i}^T, H_{d_i}^T)$ .

This gives us following:

**Corollary 2.7.** *Let  $\mathbb{F}_0$  has the property FE. Then, the number of conjugacy classes of unipotent elements in  $U(V, B)$  is finite. And hence, the number of  $z$ -classes of unipotent elements in  $U(V, B)$  is finite.*

*Proof.* In view of above Proposition let the minimal polynomial be  $(x-1)^d$  thus we have  $p(x) = x-1$ . Then the conjugacy classes correspond to a sequence  $1 \leq d_1 \leq d_2 \leq \dots \leq d_k = d$  and hermitian spaces  $\{(V_{d_1}^T, H_{d_1}^T), \dots, (V_{d_k}^T, H_{d_k}^T)\}$  up to equivalence. Now  $\underline{E}_{d_i}^T = \frac{\mathbb{F}[T]}{\langle T-1 \rangle} \cong \mathbb{F}$ . Then, by the Wall's Approximation Theorem, the number of non-equivalent hermitian forms  $(V, B)$  are exactly equal to the number of non-equivalent hermitian forms  $(\underline{V}, \underline{B})$ . However we know that there are only finitely many non-equivalent hermitian forms over  $\mathbb{F}$  as  $\mathbb{F}_0$  has property FE. Thus  $H_{d_i}^T$  has only finitely many choices for each  $i$ . Hence the result.  $\square$

### 3. $z$ -CLASSES IN UNITARY GROUPS AND FIELDS WITH PROPERTY FE

A unitary group is an algebraic group defined over  $\mathbb{F}_0$ . Since we are working with perfect fields, an element  $T \in U(V, B)$  has a Jordan decomposition,  $T = T_s T_u = T_u T_s$  where  $T_s$  is semisimple and  $T_u$  is unipotent. Further one can use this to compute centralizer  $\mathcal{Z}_{U(V, B)}(T) = \mathcal{Z}_{U(V, B)}(T_s) \cap \mathcal{Z}_{U(V, B)}(T_u)$ . So, the Jordan decomposition helps us reduce the study of conjugacy and computation of centralizer of an element to the study of that of its semisimple and unipotent parts. In this section first, we analyze semisimple elements and then we prove our main theorem.

**3.1. Semisimple  $z$ -classes.** Let  $T \in U(V, B)$  be a semisimple element. We begin with analyzing basic cases.

**Lemma 3.1.** *Let  $T \in U(V, B)$  be a semisimple element such that the minimal polynomial is either  $p(x)$  which is irreducible, self- $U$ -reciprocal or  $q(x)\tilde{q}(x)$  where  $q(x)$  is irreducible non-self- $U$ -reciprocal. Let  $E = \frac{\mathbb{F}[x]}{\langle p(x) \rangle}$  in the first case and  $\frac{\mathbb{F}[x]}{\langle q(x) \rangle}$  in the second case. Then the  $z$ -class of  $T$  is determined by the following:*

- (1) algebra  $E$  over  $\mathbb{F}$ , and,
- (2) equivalence class of  $E$ -valued hermitian form  $H^T$  on  $V^T$ .

*Proof.* Suppose  $S, T \in U(V, B)$  are in the same  $z$ -class, then  $\mathcal{Z}_{U(V, B)}(S) = g\mathcal{Z}_{U(V, B)}(T)g^{-1}$  for some  $g \in U(V, B)$ . We may replace  $T$  by its conjugate  $gTg^{-1}$ , so we get  $\mathcal{Z}_{U(V, B)}(S) = \mathcal{Z}_{U(V, B)}(T)$ , thus  $U(V^S, H^S) = U(V^T, H^T)$ . Hence  $(V^S, H^S)$  is equivalent to  $(V^T, H^T)$ . So, in particular,  $E^S$  and  $E^T$  are isomorphic as  $\mathbb{F}$ -algebra. Converse follows from Proposition 2.3.  $\square$

Now for the general case, let  $T \in U(V, B)$  be a semisimple element with minimal polynomial written as in equation 2.1

$$m_T(x) = \prod_{i=1}^{k_1} p_i(x) \prod_{j=1}^{k_2} (q_j(x) \tilde{q}_j(x)),$$

where  $p_i(x)$  are irreducible, self-U-reciprocal polynomials of degree  $m_i$  and  $q_j(x)$  are irreducible but not self-U-reciprocal of degree  $l_j$ . Let the characteristic polynomial of  $T$  be

$$\chi_T(x) = \prod_{i=1}^{k_1} p_i(x)^{d_i} \prod_{j=1}^{k_2} (q_j(x) \tilde{q}_j(x))^{r_j}.$$

Let us write the primary decomposition of  $V$  with respect to  $m_T$  into  $T$ -invariant subspaces as

$$(3.1) \quad V = \bigoplus_{i=1}^{k_1} V_i \bigoplus \bigoplus_{j=1}^{k_2} (W_j + W_j^*).$$

Denote  $E_i = \frac{\mathbb{F}[x]}{\langle p_i(x) \rangle}$  and  $K_j = \frac{\mathbb{F}[x]}{\langle q_j(x) \rangle}$  the field extensions of  $\mathbb{F}$  of degree  $m_i$  and  $l_j$  respectively.

**Theorem 3.2.** *With notation as above, let  $T \in U(V, B)$  be a semisimple element. Then the  $z$ -class of  $T$  is determined by the following:*

- (1) *A finite sequence of integers  $(m_1, \dots, m_{k_1}; l_1, \dots, l_{k_2})$  each  $\geq 1$  such that*

$$n = \sum_{i=1}^{k_1} d_i m_i + 2 \sum_{j=1}^{k_2} r_j l_j.$$

- (2) *Finite field extensions  $E_i$  of  $\mathbb{F}$  of degree  $m_i$  for  $1 \leq i \leq k_1$  and  $K_j$  of  $\mathbb{F}$  of degree  $l_j$ , for  $1 \leq j \leq k_2$ .*
- (3) *And equivalence classes of  $E_i$ -valued hermitian forms  $H_i$  of rank  $d_i$  and  $K_j \times K_j$ -valued hermitian forms  $H'_j$  of rank  $r_j$ .*

Further, with these notation,  $\mathcal{Z}_{U(V, B)}(T) \cong \prod_{i=1}^{k_1} U_{d_i}(H_i) \times \prod_{j=1}^{k_2} GL_{r_j}(K_j)$ .

*Proof.* The proof of this follows from Proposition 2.3 and Lemma 3.1. □

**Corollary 3.3.** *Let  $\mathbb{F}_0$  has property FE. Then the number of semisimple  $z$ -classes in  $U(V, B)$  is finite.*

*Proof.* This follows if we show that there are only finitely many hermitian forms up to equivalence of any degree  $n$ . We use Jacobson's theorem (see Theorem in [Ja]) that equivalence of hermitian forms  $B$  over  $\mathbb{F}$  is given by equivalence of corresponding quadratic forms  $q(x) = B(x, x)$  over  $\mathbb{F}_0$ . However because of the FE property of  $\mathbb{F}_0$  it turns



out that  $\mathbb{F}_0^*/\mathbb{F}_0^{*2}$  is finite and hence there are only finitely many quadratic forms of degree  $n$  over  $\mathbb{F}_0$ . This proves the required result.  $\square$

**3.2. Proof of the Theorem 1.2.** The number of conjugacy classes of centralizers of semisimple elements is finite follows from the Corollary 3.3. Hence, up to conjugacy, there are finitely many possibilities for  $\mathcal{Z}_{U(V,B)}(s)$  for  $s$  semisimple in  $U(V, B)$ . Let  $T \in U(V, B)$ , then it has a Jordan decomposition  $T = T_s T_u = T_u T_s$ . Recall  $\mathcal{Z}_{U(V,B)}(T) = \mathcal{Z}_{U(V,B)}(T_s) \cap \mathcal{Z}_{U(V,B)}(T_u)$  and  $T_u \in Z(T_s)^\circ$ . Now  $\mathcal{Z}_{U(V,B)}(T_s)$  is a product of certain unitary groups and general linear groups possibly over a finite extension of  $\mathbb{F}$ . Corollary 2.7 applied on the group  $\mathcal{Z}_{U(V,B)}(T_s)$  implies that, up to conjugacy,  $T_u$  has finitely many possibilities in  $\mathcal{Z}_{U(V,B)}(T_s)$ . Hence, up to conjugacy,  $\mathcal{Z}_{U(V,B)}(T)$  has finitely many possibilities in  $U(V, B)$ . Therefore the number of  $z$ -classes in  $U(V, B)$  is finite.

**3.3. Non-example.** We end this section by giving a non-example which shows that the property FE on the field is necessary to get finiteness of  $z$ -classes for classical groups. Consider  $\mathbb{F} = \mathbb{Q}[\sqrt{d}]$ , a quadratic extension. We embed  $GL_2(\mathbb{Q})$  in  $U_4$  with respect to the hermitian form  $\begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}$  given by

$$A \mapsto \begin{pmatrix} A & \\ & {}^t \bar{A}^{-1} \end{pmatrix}.$$

Now the group  $GL_2(\mathbb{Q})$  has infinitely many semisimple  $z$ -classes. Any degree 2 irreducible polynomial  $f(x) \in \mathbb{Q}[x]$  gives a companion matrix  $C_f \in GL_2(\mathbb{Q})$ . It can be checked that the centralizers of such elements are conjugate in  $GL_2(\mathbb{Q})$  if and only if their splitting fields are isomorphic which, in turn, gives infinitely many  $z$ -classes in  $U_4$ .

#### 4. COUNTING $z$ -CLASSES IN UNITARY GROUP

We recall that there could be more than one non-equivalent non-degenerate hermitian form over a given field  $\mathbb{F}$  and hence more than one non-isomorphic unitary group. In this section, we want to count the number of  $z$ -classes and write its generating function. Special focus is on the unitary group over finite field  $\mathbb{F} = \mathbb{F}_{q^2}$  of characteristic  $\neq 2$  with  $\sigma$  given by  $\bar{x} = x^q$  and  $\mathbb{F}_0 = \mathbb{F}_q$ . It is well known that over a finite field there is a unique non-degenerate hermitian form up to equivalence, thus unique unitary group up to conjugation. We denote the unitary group by  $U_n(q) = \{g \in GL_n(q^2) \mid {}^t g J \bar{g} = J\}$  where  $J$  is an invertible hermitian matrix (for example, the identity matrix). The group  $GL_n(q)$  and  $U_n(q)$  both are subgroups of  $GL_n(q^2)$ . In view of Ennola duality, the representation theory of both these groups is closely related (for example see [SV, TV]). For applications in the subject of derangements see Burness and Giudici [BG]. Thus it is always useful

to compare any computation for  $U_n(q)$  with that of  $GL_n(q)$ . We begin with recording some well known results about  $GL_n$ .

**4.1.  $z$ -classes in General Linear Group.** Let  $p(n)$  denote the number of partitions of  $n$  with generating function  $p(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ . Let  $z_k(n)$  denote the number of  $z$ -classes in  $GL_n(k)$  and the generating function be denoted by  $z_k(x)$ . Let us denote a partition of  $n$  by  $(1^{k_1} 2^{k_2} \dots n^{k_n})$  with  $n = \sum_i i k_i$ . For convenience, we denote this by  $n \vdash (1^{k_1} 2^{k_2} \dots n^{k_n})$ .

**Proposition 4.1.** *Let  $K$  be an algebraically closed field. Then,*

- (1) *the number of  $z$ -classes of semisimple elements in  $GL_n(K)$  is  $p(n)$  and is equal to the number of  $z$ -classes of unipotent elements.*
- (2) *The number of  $z$ -classes in  $GL_n(K)$  is*

$$z(n) = \sum_{n \vdash (1^{k_1} 2^{k_2} \dots n^{k_n})} \prod_{i=1}^n \binom{p(i) + k_i - 1}{k_i}.$$

*and the generating function is*

$$z(x) = z_K(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{p(i)}}.$$

This follows from the theory of Jordan canonical forms. Green computed the number of  $z$ -classes in  $GL_n(q)$  (see section 1 in [Gr]) which is the function  $t(n)$  there. We list them here in our notation.

**Proposition 4.2.** *Let  $z(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{p(i)}}$ . Then,*

- (1)  $z_{\mathbb{C}}(x) = z(x)$ .
- (2)  $z_{\mathbb{R}}(x) = z(x)z(x^2)$ .
- (3)  $z_{\mathbb{F}_q}(x) = \prod_{i=1}^{\infty} z(x^i)$ .

To compare these numbers we make a table for small rank. The last row of this table is there in the work of Green.

	$z(1)$	$z(2)$	$z(3)$	$z(4)$	$z(5)$	$z(6)$	$z(7)$	$z(8)$	$z(9)$	$z(10)$
$\mathbb{C}$	1	3	6	14	27	58	111	223	424	817
$\mathbb{R}$	1	4	7	20	36	87	162	355	666	1367
$\mathbb{F}_q$	1	4	8	22	42	103	199	441	859	1784

**4.2.  $z$ -classes in Hyperbolic Unitary Group.** In geometry the unitary groups used are over  $\mathbb{C}$ . Let  $V$  be a vector space over  $\mathbb{C}$  of dimension  $n + 1$ . Hermitian forms are classified by signature (as in the case of quadratic forms over  $\mathbb{R}$ ) and the corresponding groups are denoted as  $U(r, s)$  where  $r + s = n + 1$ . The group given by the identity matrix is compact unitary group denoted as  $U_{n+1}(I_{n+1}) = U(n + 1, 0)$ . Genus number (which is  $z$ -classes) of compact special unitary group has been computed in [Bo] (see Theorem 3.1). We record the result here as follows:

**Proposition 4.3.** *The number of  $z$ -classes in  $U_{n+1}(I_{n+1})$  is  $p(n + 1)$ .*

The  $z$ -classes of  $U(n, 1)$  has been discussed by Cao and Gongopadhyay in [CG]. Here we present the number of  $z$ -classes in this group using the parametrization described there. Recall that the hermitian matrix used there is  $\beta = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$  and the unitary group is  $U(n, 1) = \{g \in GL(n + 1, \mathbb{C}) \mid {}^t \bar{g} \beta g = \beta\}$ . The elements of this group are classified as either elliptic, hyperbolic or parabolic depending on their fixed points. Using conjugation classification we know that if an element  $g \in U(n, 1)$  is elliptic or hyperbolic, then they are always semisimple. But a parabolic element need not be semisimple. However it has a Jordan decomposition  $g = g_s g_u$ , where  $g_s$  is elliptic, hence semisimple, and  $g_u$  is unipotent. In particular if a parabolic isometry is unipotent, then it has minimal polynomial  $(x - 1)^2$  or  $(x - 1)^3$  and is called vertical translation or non-vertical translation respectively.

**Proposition 4.4.** *In the group  $U(n, 1)$ ,*

- (1) *the number of  $z$ -classes of elliptic elements is  $\sum_{m=1}^{n+1} p(n + 1 - m)$ .*
- (2) *The number of  $z$ -classes of hyperbolic elements is  $p(n - 1)$ .*
- (3) *The number of  $z$ -classes of parabolic elements is  $2 + p(n - 1) + p(n - 2)$ , in this case  $n \geq 2$ .*

*Proof.* Let  $T \in U(n, 1)$  be an elliptic element. Then  $T$  has a negative class of eigenvalue say  $[\lambda]$ . Let  $m = \dim(V_\lambda)$  which is  $\geq 1$ . It follows from the conjugacy classification that all the eigen values have norm 1, and there is a negative eigenvalue. All other eigenvalues are of positive type. Then  $V = V_\lambda \perp V_\lambda^\perp = V_\lambda \perp (\perp_{i=1}^s V_{\lambda_i})$ . Suppose  $\dim(V_{\lambda_i}) = r_i$ , then  $\mathcal{Z}_{U(n, 1)}(T) = \mathcal{Z}_{U(V_\lambda)}(T|_{V_\lambda}) \times \prod_{i=1}^s U(r_i)$ . Now since  $T|_{V_\lambda}$  is of negative type, so  $\mathcal{Z}(T|_{V_\lambda}) = U(m - 1, 1)$ . Here  $n + 1 = m + \sum_{i=1}^s r_i$ , therefore  $\sum_{i=1}^s r_i = n + 1 - m$ . This gives that the number of  $z$ -classes of elliptic elements is  $\sum_{m=1}^{n+1} p(n + 1 - m)$ .

Now, suppose  $T \in U(n, 1)$  is hyperbolic. Then  $V$  has an orthogonal decomposition as  $V = V_r \perp (\perp_{i=1}^k V_i)$  where  $\dim(V_i) = r_i$  and  $V_i$  is the eigenspace of  $T$  corresponding to the similarity class of positive eigenvalue  $[\lambda_i]$  with  $|\lambda_i| = 1$ . The subspaces  $V_r$  are two dimensional  $T$ -invariant subspaces spanned by the corresponding similarity class of

null-eigenvalues  $[re^{i\theta}], [r^{-1}e^{i\theta}]$  for  $r > 1$ , respectively. Then  $\mathcal{Z}_{U(n,1)}(T) = \mathcal{Z}(T|_{V_r}) \times \prod_{j=1}^k U(r_j) = S^1 \times \mathbb{R} \times \prod_{j=1}^k U(r_j)$ . Here  $n+1 = 2 + \sum_{j=1}^k r_j$ , i.e.,  $\sum_{j=1}^k r_j = n-1$ . Thus, the number of  $z$ -classes of hyperbolic elements is  $p(n-1)$ .

Let  $T \in U(n,1)$  be parabolic. First, let  $T$  be unipotent. If the minimal polynomial of  $T$  is  $(x-1)^2$ , i.e.,  $T$  is a vertical translation then  $\mathcal{Z}_{U(n,1)}(T) = U(n-1) \ltimes (\mathbb{C}^{n-1} \times \mathbb{R})$ . If the minimal polynomial of  $T$  is  $(x-1)^3$ , i.e.,  $T$  is non-vertical translation then  $\mathcal{Z}_{U(n,1)}(T) = (S^1 \times U(n-2)) \ltimes ((\mathbb{R} \times \mathbb{C}^{n-2}) \ltimes \mathbb{R})$ . Hence there are exactly two  $z$ -classes of unipotents, one corresponds to the vertical translation and the other to the non-vertical translation. Now assume that  $T$  is not unipotent. Suppose that the similarity class of null-eigenvalue is  $[\lambda]$ . Then  $V$  has a  $T$ -invariant orthogonal decomposition as  $V = V_\lambda \perp V_\lambda^\perp$ , where  $V_\lambda$  is a time like  $T$ -indecomposable subspace of  $\dim(V_\lambda) = m$  which is either 2 or 3. Then  $\mathcal{Z}_{U(n,1)}(T) = \mathcal{Z}(T|_{V_\lambda}) \times \mathcal{Z}(T|_{V_\lambda^\perp})$ . For each choice of  $\lambda$ , there are exactly one choice for the  $z$ -classes of  $T|_{V_\lambda}$  in  $U(m-1,1)$ , i.e.,  $U(1,1)$  or  $U(2,1)$ . Note that  $T|_{V_\lambda^\perp}$  can be embedded into  $U(n+1-m)$ . Hence it suffices to find out the number of  $z$ -classes of  $T|_{V_\lambda}$  in  $U(m-1,1)$ . Hence the total number of  $z$ -classes of non-unipotent parabolic is  $p(n-1) + p(n-2)$ . Therefore the total number of  $z$ -classes of parabolic transformations is  $2 + p(n-1) + p(n-2)$  ( $n \geq 2$ ).  $\square$

**4.3.  $z$ -classes in Finite Unitary Group.** Now let us look at the finite unitary group in characteristic  $\neq 2$ .

**Proposition 4.5.** (1) *The number of  $z$ -classes of unipotent elements in  $U_n(q)$  is  $p(n)$  which is equal to the number of  $z$ -classes of unipotent elements in  $GL_n(q)$ .*  
 (2) *The number of  $z$ -classes of semisimple elements in  $U_n(q)$  is equal to the number of  $z$ -classes of semisimple elements in  $GL_n(q)$ .*

*Proof.* Let  $u = [J_1^{a_1} J_2^{a_2} \dots J_n^{a_n}]$  be a unipotent element in  $GL_n(q^2)$  written in Jordan block form. Wall proved the following membership test (see Case(A) on page 34 of [Wa]). Let  $A \in GL_n(q^2)$  then  $A$  is conjugate to  ${}^t \bar{A}^{-1}$  in  $GL_n(q^2)$  if and only if  $A$  is conjugate to an element of  $U_n(q)$ . Since unipotents are conjugate to its own inverse in  $GL_n(q^2)$ , this implies  $u$  is conjugate to  ${}^t \bar{u}^{-1}$  in  $GL_n(q^2)$ . Hence  $u$  is conjugate to an element of  $U_n(q)$ . Wall also proved that two elements of  $U_n(q)$  are conjugate in  $U_n(q)$  if and only if they are conjugate in  $GL_n(q^2)$  (see also 6.1 [Ma]). Thus, up to conjugacy, there is a one-one correspondence of unipotent elements between  $GL_n(q^2)$  and  $U_n(q)$ . This gives that the number of unipotent conjugacy classes in  $U_n(q)$  is  $p(n)$  and is same as that of  $GL_n(q)$ . Now, we note that  $\mathcal{Z}_{U_n(q)}(u) = Q \prod_{i=1}^n U_{a_i}(q)$  where  $Q = R_u(\mathcal{Z}_{U_n(q)}(u))$  and  $|Q| = q^{\sum_{i=2}^n (i-1)a_i^2 + 2 \sum_{i < j} i a_i a_j}$  (see Lemma 3.3.8 [BG]). Clearly the centralizers are distinct and thus the number of unipotent  $z$ -classes in  $U_n(q)$  is  $p(n)$ .

For semisimple elements we use the Theorem 3.2. Over a finite field, we get that semisimple  $z$ -classes are characterized by simply  $n = \sum_{i=1}^{k_1} d_i m_i + \sum_{j=1}^{k_2} e_j l_j$  where  $d_i$  is

odd (being degree of monic, irreducible, self-U-reciprocal polynomial, see Proposition 2.1) and  $e_j = 2r_j$  is even. This corresponds to the number of ways  $n$  can be written as  $n = \sum_i e_i f_i$  which is same as the number of semisimple  $z$ -classes in  $GL_n(q)$ .  $\square$

**4.4. Proof of Theorem 1.3.** Recall that if  $g = g_s g_u$  is a Jordan decomposition of  $g$  then  $\mathcal{Z}_{U_n(q)}(g) = \mathcal{Z}_{U_n(q)}(g_s) \cap \mathcal{Z}_{U_n(q)}(g_u) = \mathcal{Z}_{\mathcal{Z}_{U_n(q)}(g_s)}(g_u)$ , and the structure of  $\mathcal{Z}_{U_n(q)}(g_s)$  in the Theorem 3.2 implies that

$$\text{the number of } z\text{-classes in } U_n(q) = \sum_{[s]_z} \text{number of unipotent } z\text{-classes in } \mathcal{Z}_{U_n(q)}(s)$$

where the sum runs over semisimple  $z$ -classes. Hence the number of  $z$ -classes in  $U_n(q)$  is same as the number of  $z$ -classes in  $GL_n(q)$ .

## 5. FURTHER PLAN

Given wide interest and application in group theory it is interesting to compute centralizers and  $z$ -classes in algebraic groups over base field. We plan to continue our study for other groups specially for exceptional groups. We hope that representation theorists will find this work useful as did Green.

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